Expected Utility Over Money and Risk Aversion

Econ 3030 Fall 2025

Lecture 11

Outline

Probability Distribution On Wealth

- Many applications of expected utility consider preferences on probability distributions of wealth (a continuous variable).
- A probability distribution is characterized by its cumulative distribution function.

Definition

A cumulative distribution function (cdf) $F : \mathbb{R} \rightarrow [0, 1]$ satisfies:

- $x \geq y$ implies $F(x) \geq F(y)$ (nondecreasing);
- $\lim_{y \downarrow x} F(y) = F(x)$ (right continuous);[a]
- $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow \infty} F(x) = 1$.

[a] Recall that, if it exists, $\lim_{y \downarrow x} f(y) = \lim_{n \rightarrow \infty} f(x + \frac{1}{n})$.

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Notation

- μ_F denotes the expected value of F, i.e. $\mu_F = \int x \, dF(x)$.
- δ_x is the degenerate distribution function at x; i.e. δ_x yields x with certainty:

$$\delta_x(z) = \begin{cases} 0 & \text{if } z < x \\ 1 & \text{if } z \geq x \end{cases}.$$

Expected Utility Of Wealth

- As usual, the space of all distribution functions is convex and one can define preferences on it.

The utility index $v : \mathbb{R} \to \mathbb{R}$ is defined over wealth (NOTE: values can be negative).

- The expected utility is the integral of v with respect to F

$$\int v(x) dF(x) = \int v dF$$

  - If F is differentiable, the expectation is computed using the density $f = F'$: $\int v(x) dF = \int v(x) f(x) dx$.

von Neumann and Morgenstern Expected Utility

Under some axioms, there exists a utility function U on distributions defined as $U(F) = \int v(x)\, dF$, for some continuous index $v : \mathbb{R} \to \mathbb{R}$ over wealth, such that

$$F \succsim G \iff \int v(x)\, dF \geq \int v(x)\, dG$$

- Axioms are not important (need a stronger continuity assumption).
- We always think of v as a weakly increasing function (more wealth cannot be bad).

Simple Probability

- A simple probability distribution π on $X \subset \mathbb{R}$ is specified by
  - a finite subset of X called the support and denoted $supp(\pi)$, and
  - for each $x \in X$, $\pi(x) > 0$ with $\sum_{x \in supp(\pi)} \pi(x) = 1$
- If we restrict attention to simple probability distributions, then even if X is infinite, only elements with strictly positive probability count.

- The utility index $v : X \to \mathbb{R}$ is defined over wealth (can be negative).
- The expected utility is the expected value of v with respect to π

$$\sum_{x \in supp(\pi)} \pi(x) v(x)$$

- One can write more money is better as: for each $x, y \in X$ such that $x > y$ then $\delta_x \succ \delta_y$.
- We can use this setting to think about many applied choice under uncertainty problems like betting and insurance.

Betting

A Gamble

- Suppose an individual is offered the following bet:

win ax with probability p	lose x with probability $1-p$

- The expected value of this bet is

$$pax + (1-p)(-x) = [pa + (1-p)(-1)]\,x$$

Definition

A bet is actuarially fair if it has expected value equal to zero (i.e. $a = \frac{1-p}{p}$); it is better than fair if the expected value is positive and worse than fair if it is negative.

How does she evaluate this bet? Use the expected utility model to find out

- If vNM index is $v(\cdot)$ and initial wealth is w, expected utility is:

$$\overbrace{p}^{\text{probability of winning}}\ \underbrace{v(w+ax)}_{\text{utility of wealh if win}} + \overbrace{(1-p)}^{\text{probability of losing}}\ \underbrace{v(w-x)}_{\text{utility of wealth if lose}}$$

Betting and Expected Utility

How much does she want of this bet? Answer by finding the optimal x.

win ax with probability p	lose x with probability $1-p$

- The consumer solves
$$\max_x pv(w+ax)+(1-p)v(w-x)$$
- The FOC is
$$pav'(w+ax)=(1-p)v'(w-x)$$
  - rearranging
$$\frac{pa}{(1-p)}=\frac{v'(w-x)}{v'(w+ax)}$$
- If the bet is fair, the left hand side is 1. Therefore, at an optimum, the right hand side must also be 1.
- If the vNM utility function is strictly increasing and strictly concave ($v'>0$ and $v''<0$), the only way a fair bet can satisfy this FOC is to solve
$$w+ax=w-x$$
which implies $x=0$.
- She will take no part of a fair bet.
- What happens with a better than fair bet?

An Insurance Problem

- An individual faces a potential "accident":

the loss is L with probability π	nothing happens with probability $1-\pi$

Definition

An insurance contract establishes an initial premium P and then reimburses an amount Z if and only if the loss occurs.

Definition

Insurance is actuarially fair when its expected cost is zero; it is less than fair when its expected cost is positive.

- The expected cost (to the individual) of an insurance contract is

$$P - \left[\underbrace{\pi(-Z)}_{\text{loss}} + \underbrace{(1-\pi)\ (0)}_{\text{no loss}}\right] = P - \pi Z$$

- Fair insurance means

$$P = \pi Z$$

An Insurance Problem

An individual with current wealth W and utility function $v(\cdot)$ faces a potential accident:

$\boxed{\text{lose } L \text{ with probability } \pi}$ or $\boxed{\text{lose zero with probability } 1-\pi}$

- If she buys insurance, her expected utility is

$$\pi v(\underbrace{W - L - P + Z}_{\text{wealth if loss}}) + (1-\pi)\, v(\underbrace{W - P}_{\text{wealth if no loss}})$$

  - For example, if the loss is fully reimbursed ($Z = L$), this becomes
$$\pi v\,(W - P) + (1-\pi)\, v\,(W - P) = v\,(W - P)$$

- Will she buy any insurance? Yes if

$$\underbrace{\pi v\,(W - L - P + Z) + (1-\pi)\, v\,(W - P)}_{\text{expected utility with insurance}} \geq \underbrace{\pi v\,(W - L) + (1-\pi)\, v\,(W)}_{\text{expected utility without insurance}}$$

- Find how much coverage she wants (if any) by finding the optimal Z.
- The answer depends on the premium set by the insurance company P (which could depend on Z) as well as the curvature of the utility function v.

Curvature of the Utility Function

The answers to the previous problems depend on the curvature of the utility function v.

- The curvature of v captures important characteristics of preferences in many applied situations.

Risk Aversion

Definitions

The preference relation $\succsim$ is

- risk averse if, for all cumulative distribution functions F, $\delta_{\mu_F} \succsim F$.
- risk loving if, for all cumulative distribution functions F, $F \succsim \delta_{\mu_F}$.
- risk neutral if it is both risk averse and risk loving: $\delta_{\mu_F} \sim F$.

- DM is risk averse if she always prefers the expected value μ_F for sure to the uncertain distribution F.
- This definition does not depend on the expected utility representation (or any other).

Remark

Risk attitudes are defined directly from preferences.

Risk Aversion: An example

Exercise

Let $\succsim$ be a preference relation on the space of all cumulative distribution functions represented by the following utility function:

$$U(F) = \begin{cases} x & \text{if } F = \delta_x \text{ for some } x \in \mathbf{R} \\ 0 & \text{otherwise} \end{cases}$$

- True of false: $\succsim$ is risk averse.
  - **False**: If $\mu_F < 0$, then $F \succ \mu_F$.

Certainty Equivalent

Definition

Given a strictly increasing and continuous vNM index v over wealth, the certainty equivalent (CE) of F, denoted $c(F, v)$, is defined by

$$v(c(F, v)) = \int v(\cdot)\, dF.$$

- The certainty equivalent of F is the amount of wealth $c(\cdot)$ such that $c(\cdot) \sim F$.
  - DM is indifferent between a distribution and the certainty equivalent of that distribution.
  - The certainty equivalent is constructed to satisfy this indifference.
- One can compare two lotteries by comparing their certainty equivalents.
- Unlike risk aversion, the certainty equivalent definition assumes a given preference representation (needs some utility function that represents preferences).
- The value of the certainty equivalent is related to risk aversion.

Risk Premium

Definition

Given a strictly increasing and continuous vNM index v over wealth, the risk premium of F, denoted $r(F, v)$ is defined by

$$r(F, v) = \mu_F - c(F, v).$$

- This measures the difference between the expected value of a particular distribution and its certainty equivalent.
- The definition of risk premium also assumes a given preference representation.
- The risk premium is also related to risk aversion.

Risk Aversion, Certainty Equivalent, and Risk Premium

- If preferences satisfy the vNM axioms, risk aversion is characterized by concavity of the utility index and a non-negative risk-premium.

Proposition

Suppose $\succsim$ has an expected utility representation and v is the corresponding von Neumann and Morgestern utility index over money.The following are equivalent:

1. $\succsim$ *is risk averse;*
2. v *is concave;*
3. $r(F, v) \geq 0$*;*

- The proof uses Jensen's inequality.

Jensen's Inequality

Reminder: *a function* $f : \mathbb{R} \to \mathbb{R}$ *is concave if for any* $x, y \in \mathbb{R}$ *and any* $\alpha \in [0,1]$

$$f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha) f(y)$$

Jensen's inequality

A function g is concave if and only if for all distributions F

$$g\left(\int x dF\right) \geq \int g(x)\, dF$$

- This says

$$g(\mathbf{E}(X)) \geq \mathbf{E}(g(X))$$

Consequences of Jensen's inequality

- Hence, $v(\cdot)$ is concave if and only if for all distributions F

$$\underbrace{v\left(\int dF\right)}_{\text{utility of the expected value of } F} \geq \underbrace{\int v dF}_{\text{expected utility of } F}$$

Risk Aversion, CE, and Risk Premium

$$\underbrace{\succsim \text{ is risk averse}}_{(1)} \Leftrightarrow \underbrace{v \text{ is concave}}_{(2)} \Leftrightarrow \underbrace{r(F,v) \geq 0}_{(3)}$$

We prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). Start with (1) $\Rightarrow$ (2).

Proof.

$\succsim$ is risk averse, hence $\delta_{\mu_F} \succsim F$ for all $F \in \Delta\mathbb{R}$.

- For any $x, y \in \mathbb{R}$ and $\alpha \in [0,1]$, let the discrete random variable X be such that $P(X = x) = \alpha$ and $P(X = y) = 1 - \alpha$. Let $F^{\alpha}_{x,y}$ be the associated cumulative distribution.
- By risk aversion we have:

$$\begin{aligned} v(\mu_{F^{\alpha}_{x,y}}) &\geq \int v(z) dF^{\alpha}_{x,y}(z) \\ &\Rightarrow \\ v(\alpha x + (1-\alpha)y) &\geq \sum_z v(z) P(X = z) = \alpha v(x) + (1-\alpha) v(y) \end{aligned}$$

- Thus v is concave.

□

Risk Aversion, CE, and Risk Premium

$$\underbrace{\succsim \text{ is risk averse}}_{(1)} \Leftrightarrow \underbrace{v \text{ is concave}}_{(2)} \Leftrightarrow \underbrace{r(F, v) \geq 0}_{(3)}$$

Now prove that (2) $\Rightarrow$ (3)

Proof.

Let v be concave, and X be a random variable with cdf F.

- By Jensen's inequality:

  $$v(\mathbf{E}(X)) \geq \mathbf{E}(v(X))$$

  or

  $$v(\mu_F) \geq \int v(x)dF(x) = v(c(F, v))$$

- Since v is an increasing function, we have

  $$\mu_F \geq c(F, v)$$

- Thus

  $$\mu_F - c(F, v) = r(F, v) \geq 0$$

□

Risk Aversion, CE, and Risk Premium

$$\underbrace{\succsim \text{ is risk averse}}_{(1)} \Leftrightarrow \underbrace{v \text{ is concave}}_{(2)} \Leftrightarrow \underbrace{r(F,v) \geq 0}_{(3)}$$

$(3) \Rightarrow (1)$

Proof.

Let $r(F,v) \geq 0$ for all cdfs F.

- Then we have
$$\mu_F \geq c(F,v)$$
- which in turn implies that
$$v(\mu_F) \geq v(c(F,v)) = \int v(x)dF(x)$$
- Hence $\delta_{\mu_F} \succsim F$ for all F; therefore $\succsim$ is risk averse. □

We have shown that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$, thus the proof is complete.

Next Class

- Relative Risk Aversion
- Stochastic Dominance
- Random Consumption Bundles